

PLANE OVERPARTITIONS AND CYLINDRIC PARTITIONS

SYLVIE CORTEEL[!], CYRILLE SAVELIEF[!] AND MIRJANA VULETIĆ^{*}

ABSTRACT. Generating functions of plane overpartitions are obtained using various methods: non-intersecting paths, RSK type algorithms and symmetric functions. We extend some of the results to cylindric partitions. Also, we show that plane overpartitions correspond to domino tilings and we give some basic properties of this correspondence.

1. INTRODUCTION

The goal of the first part of this paper is to introduce a new object: the plane overpartitions and to give several enumeration formulas for these plane overpartitions. A *plane overpartition* is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all the others are overlined. An example of a plane overpartition is

$$\begin{array}{cccc} 4 & 4 & \bar{4} & \bar{3} \\ \bar{4} & 3 & 3 & \bar{3} \\ \bar{4} & \bar{3} & & \\ 3 & & & \end{array}$$

This is an overpartition of the shape $(4, 4, 2, 1)$, with the weight equal to 38 and 6 overlined parts.

This paper takes its place in the series of papers on overpartitions started by Corteel and Lovejoy [CL]. The motivation is to show that the generating function of plane overpartitions is :

$$\prod_{n \geq 1} \left(\frac{1 + q^n}{1 - q^n} \right)^n.$$

In this paper, we give several proofs of this result and several refinements and generalizations. Namely, we give

Result 1. *the hook-length formula for the generating function of plane overpartitions of a given shape, see Theorem 3;*

Result 2. *the hook formula for the generating function for reverse plane overpartitions, see Theorem 5;*

Result 3. *the generating formula for plane overpartitions with bounded parts, see Theorem 6.*

The goal of the second part of this paper is to extend the generating formula for cylindric partitions due to Borodin and the 1-parameter generalized MacMahon's formula due to Vuletić:

$$(1.1) \quad \sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} A_{\Pi}(t) q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1 - tq^n}{1 - q^n} \right)^n,$$

where the weight $A_{\Pi}(t)$ is a polynomial in t that we describe below.

Given a plane partition Π , we decompose each connected component (a connected set of boxes filled with a same number) of its diagram into border components (i.e. rim hooks) and assign to each border component a level which is its diagonal distance to the end of the component. We associate to each border component of level i , the weight $(1 - t^i)$. The weight of the plane partition Π is $A_{\Pi}(t)$ the product of the weights of its border components. See [V2] for further details and Figure 1 for an example of a plane partition of weight $(1 - t^{10})(1 - t^2)^2(1 - t^3)^2$.

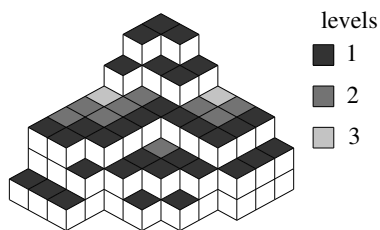


FIGURE 1. Weight of a plane partition

We give a new proof of the 1-parameter generalized MacMahon's formula. We also extend this formula to two more general objects: skew plane partitions and cylindric partitions. We give

Result 4. *1-parameter generalized formula for skew plane partitions, see Theorem 7;*

Result 5. *1-parameter generalized formula for cylindric partitions, see Theorem 8.*

In the rest of this section we give definitions and explain our results in more detail.

A partition λ is a nonincreasing sequence of positive integers $(\lambda_1, \dots, \lambda_k)$. The λ_i s are called parts of the partition and the number of parts is denoted by $\ell(\lambda)$. The weight $|\lambda|$ of λ is the sum of its parts. A partition λ can be graphically represented by the Ferrers diagram that is a diagram formed of $\ell(\lambda)$ left justified rows, where the i^{th} row consists of λ_i cells (or boxes). The conjugate of a partition λ , denoted with λ' , is a partition that has the Ferrers diagram equal to the transpose of the Ferrer diagram of λ . For a cell (i, j) of the Ferrers diagram of λ the hook length of this cell is $h_{i,j} = \lambda_i + \lambda'_j - i - j + 1$ and the content $c_{i,j} = j - i$. It is well known that the generating function of partitions that have at most n parts is $1/(q)_n$, where $(a)_n = (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$. More definitions on partitions can be found, for example, in [A] or [Mac].

An *overpartition* is a partition where the last occurrence of an integer can be overlined [CL]. Last occurrences in an overpartition are in one-to-one correspondence with corners of the Ferrers diagram and overlined parts can be represented by marking the corresponding corners. The generating function of overpartitions that have at most n parts is $(-q)_n/(q)_n$.

Let λ be a partition. A *plane partition* of shape λ is a filling of cells of the Ferrers diagram of λ with positive integers that form a nonincreasing sequence along each row and each column. We denote the shape of a plane partition Π with $\text{sh}(\Pi)$ and the sum of all entries with $|\Pi|$, called the weight of Π . It is well known, under the name of MacMahon's formula, that the generating function of plane partitions is

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^n.$$

One way to prove this is to construct a bijection between plane partitions and pairs of semi-standard reverse Young tableaux of a same shape and to use the inverse of the RSK algorithm, that gives a bijection between those pairs of those tableaux [BK].

Recall that a plane overpartition is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all the others are overlined. This definition implies that the entries strictly decrease along diagonals. Therefore a plane overpartition is a plane partition made of rim hooks (connected skew shapes containing no 2×2 block of squares) where some entries are overlined. More

precisely, it is easy to check that inside a rim hook only one entry can be chosen to be overlined or not and this entry is the upper right entry.

Plane overpartitions are therefore in bijection with rim hook plane partitions where each rim hook can be overlined or not (or weighted by 2). Recently, those weighted rim hook plane partitions were studied in [FW, FW1, V1, V2]. The first result obtained was the shifted MacMahon's formula that says that the generating function of plane overpartitions is

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane overpartition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n.$$

This was obtained as a limiting case of the generating formula for plane overpartitions which fit into a $r \times c$ box, i.e. whose shapes are contained in the rectangular shape with r rows and c columns.

Theorem 1. [FW, V1] *The generating function of plane overpartitions which fit in a $r \times c$ box is*

$$\prod_{i=1}^r \prod_{j=1}^c \frac{1+q^{i+j-1}}{1-q^{i+j-1}}.$$

This theorem was proved in [FW, V1] using Schur P and Q symmetric functions and a suitable Fock space. In [V2] the theorem was proved in a bijective way where a RSK-type algorithm (due to Sagan [Sa], see also Chapter XIII of [HH]) was used to construct a bijection between plane overpartitions and matrices of nonnegative integers where positive entries can be overlined.

Note that plane overpartitions are also in bijection with super semi-standard tableaux [Kr0] using the *jeu de taquin* [Sav].

In Section 2, we give a mostly combinatorial proof of the generalized MacMahon formula [V2]. Namely, we prove

Theorem 2. [V2]

$$(1.2) \quad \sum_{\Pi \in \mathcal{P}(r,c)} A_{\Pi}(t) q^{|\Pi|} = \prod_{i=1}^r \prod_{j=1}^c \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}.$$

where $\mathcal{P}(r, c)$ is the set of plane partitions with at most r rows and c columns. When we set $t = -1$, only the border components of level 1 have a non-zero weight and we get back Theorem 1.

The main result of Section 3 is a hook-content formula for the generating function of plane overpartitions of a given shape. More generally, we give a weighted generating function where overlined parts are weighted by some parameter a . Throughout Sections 3, 4 and 5 we assume these weights, unless otherwise stated.

Let $\mathcal{S}(\lambda)$ be the set of all plane overpartitions of shape λ . The number of overlined parts of an overpartition Π is denoted with $o(\Pi)$.

Theorem 3. *Let λ be a partition. The weighted generating function of plane overpartitions of shape λ is*

$$(1.3) \quad \sum_{\Pi \in \mathcal{S}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{\sum_i i\lambda_i} \prod_{(i,j) \in \lambda} \frac{1 + aq^{c_{i,j}}}{1 - q^{h_{i,j}}}.$$

We also give the weighted generating formula for plane overpartitions “bounded” by λ , where by that we mean plane overpartitions such that the i^{th} row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts. Let $\mathcal{B}(\lambda)$ be the set of all such plane overpartitions.

Theorem 4. *Let λ be a partition. The weighted generating function of plane overpartitions such that the i^{th} row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts is*

$$(1.4) \quad \sum_{\Pi \in \mathcal{B}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{\sum_i (i-1)\lambda_i} \prod_{(i,j) \in \lambda} \frac{1 + aq^{c_{i,j}+1}}{1 - q^{h_{i,j}}}.$$

where $h_{i,j} = \lambda_i - i + 1 + \lambda'_j - j$ is the hook length of the cell (i, j) and $c_{i,j} = j - i$ is the content of the cell (i, j) .

Note that it is enough to assign weights to overlined (or nonoverlined) parts only because generating functions where overlined and nonoverlined parts are weighted by a and b , respectively, follow trivially from the above formulas.

We prove these two theorems using a correspondence between plane overpartitions and sets of nonintersecting paths that use three kinds of steps. The work of Brenti used similar paths to compute super Schur functions [Br]. Using Gessel–Viennot results [GV] we obtain determinantal formulas for the generating functions above. Then we evaluate the determinants to obtain the hook-content formulas for these determinants. We use a simple involution to show that the Stanley hook-content formula (Theorem 7.21.2 of [Sta]) follows from our formula.

We connect these formulas to some special values of symmetric functions that are not obtained by evaluations.

The end of Section 3 is devoted to *reverse* plane overpartitions. A reverse plane (over)partition is defined like a plane (over)partition with nonincreasing property substituted with nondecreasing property and allowing 0's as entries. Precisely, a reverse plane partition of shape λ is a filling of cells of the Ferrers diagram of λ with nonnegative integers that form a nondecreasing sequence along each row and each column. A reverse plane overpartition is a reverse plane partition where only positive entries can be overlined and in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of a positive integer can be overlined or not and all others (if positive) are overlined. An example of a reverse plane overpartition is

$$\begin{array}{cccccc} 0 & 0 & 3 & 4 & 4 & \bar{4} \\ 0 & 0 & 4 & \bar{4} & & \\ 1 & \bar{3} & & & & \\ 3 & \bar{3} & & & & \end{array}$$

It was proved by Gansner [G] that the generating function of reverse plane partitions of a given shape λ is

$$(1.5) \quad \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{h_{i,j}}}.$$

Let $\mathcal{S}^R(\lambda)$ be the set of all reverse plane overpartitions of shape λ . The generating function of reverse plane overpartitions is given by the following hook-formula.

Theorem 5. *Let λ be a partition. The generating function of reverse plane overpartitions of shape λ is*

$$\sum_{\Pi \in \mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j) \in \lambda} \frac{1 + q^{h_{i,j}}}{1 - q^{h_{i,j}}}.$$

We construct a bijection between reverse plane overpartitions of a given shape and sets of nonintersecting paths whose endpoints are not fixed. Using results of [Ste] we obtain a Pfaffian formula for the generating function of reverse plane partitions of a given shape. We further evaluate the Pfaffian and obtain a bijective proof of the hook formula. When λ is the partition with r parts equal to c , this result is the generating formula for plane overpartitions fitting in a box $r \times c$, namely Theorem 1.

In Section 4 we make a connection between plane overpartitions and domino tilings. We give some basic properties of this correspondence, like how removing a box or changing a mark to unmark changes the

In Section 5 we propose a bijection between matrices and pairs of plane overpartitions based on Berele and Remmel [BeR] which gives a bijection between matrices and pairs of (k, l) -semistandard tableaux. This bijection is based on the jeu de taquin. We give another stronger version of the shifted MacMahon's formula, as we give a weighted generating function of plane overpartitions with bounded entries. Let $\mathcal{L}(n)$ be the set of all plane overpartitions with the largest entry at most n .

$$\sum_{\Pi \in \mathcal{L}(n)} a^{o(\Pi)} q^{|\Pi|} = \frac{\prod_{i,j=1}^n (1 + aq^{i+j})}{\prod_{i=1}^n \prod_{j=0}^{i-1} (1 - q^{i+j})(1 - aq^{i+j})}.$$

The image displays a complex geometric pattern composed of interlocking triangles. Each triangle contains a number, and the pattern is symmetrical. The numbers are arranged in a way that suggests a mathematical or logical sequence. The pattern is composed of several interconnected shapes, including triangles and diamonds, creating a complex, abstract design.

the i^{th} diagonal represents λ^i and if the first part of λ^i is placed at (i, j) then the first part of λ^{i+1} is placed at $(i+1, j-1)$ if $A_i = 1$ or $(i+1, j+1)$ otherwise. Observe that the profile A defines the upper border of the

diagram (part of the border with a positive slope corresponds to 0's and with negative to 1's).

A (skew) plane partition and cylindric partition are examples of interlacing sequences. A plane partition can be written as $\Lambda = (\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)$ with profile $A = (0, 0, \dots, 0, 1, \dots, 1, 1)$ and λ^i 's are diagonals of the plane partition. A skew plane partition is an interlacing sequence $\Lambda = (\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)$ with a profile $A = (A_0, A_1, \dots, A_{T-1}, A_T)$, where $A_0 = 0$ and $A_T = 1$. A cylindric partition is an interlacing sequence $\Lambda = (\lambda^0, \lambda^1, \dots, \lambda^T)$ where $\lambda^0 = \lambda^T$, and T is called the period of Λ . A cylindric partitions can be represented by the *cylindric diagram* that is obtained from the ordinary diagram by identification of the first and last diagonal.

A *connected component* of an interlacing sequence Λ is the set of all rookwise connected boxes of its diagram that are filled with a same number. We denote the number of connected components of Λ with $k(\Lambda)$. For the example from Figure 2 we have $k(\Lambda) = 19$ and its connected components are shown in Figure 2 (bold lines represent boundaries of these components).

If a box (i, j) belongs to a connected component C then we define its *level* $\ell(i, j)$ as the smallest positive integer such that $(i + \ell, j + \ell)$ does not belong to C . In other words, a level represents the distance from the “rim”, distance being measured diagonally. A *border component* is a rookwise connected subset of a connected component where all boxes have the same level. We also say that this border component is of this level. All border components are rim hooks. For the example above, border components and their levels are shown in Figure 3.

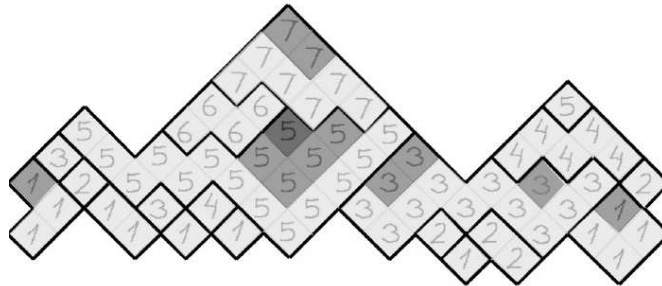


FIGURE 3. Border components with levels

For a cylindric partition we define *cylindric connected components* and *cylindric border components* in the same way but connectedness is understood on the cylinder, i.e. boxes are connected if they are rookwise connected in the cylindric diagram.

Let (n_1, n_2, \dots) be a sequence of nonnegative integers where n_i is the number of i -level border components of Λ . We set

$$(1.6) \quad A_\Lambda(t) = \prod_{i \geq 1} (1 - t^i)^{n_i}.$$

For the example above $A_\Lambda(t) = (1 - t)^{19}(1 - t^2)^6(1 - t^3)$.

Similarly, for a cylindric partition Π we define

$$A_\Pi^{\text{cyl}}(t) = \prod_{i \geq 1} (1 - t^i)^{n_i^{\text{cyl}}},$$

where n_i^{cyl} is the number of cylindric border components of level i .

In Section 6 we give a generating formula for skew plane partitions. Let $\text{Skew}(T, A)$ be the set of all skew plane partitions $\Lambda = (\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)$ with profile $A = (A_0, A_1, \dots, A_{T-1}, A_T)$, where $A_0 = 0$ and $A_T = 1$.

Theorem 7. (*Generalized MacMahon's formula for skew plane partitions; Hall-Littlewood's case*)

$$\sum_{\Pi \in \text{Skew}(T, A)} A_\Pi(t) q^{|\Pi|} = \prod_{\substack{0 \leq i < j \leq T \\ A_i=0, A_j=1}} \frac{1 - tq^{j-i}}{1 - q^{j-i}}.$$

Note that as profiles are words in $\{0, 1\}$, a profile $A = (A_0, \dots, A_T)$ encodes the border of a Ferrers diagram λ . Skew plane partitions of profile A are in one-to-one correspondence with reverse plane partitions of shape λ . Moreover, one can check that

$$\prod_{\substack{0 \leq i < j \leq T \\ A_i=0, A_j=1}} \frac{1 - tq^{j-i}}{1 - q^{j-i}} = \prod_{(i,j) \in \lambda} \frac{1 - tq^{h_{i,j}}}{1 - q^{h_{i,j}}}.$$

Therefore the Theorem of Gansner (equation (1.5)) is Theorem 7 with $t = 0$ and our Theorem 5 on reverse plane overpartitions is Theorem 7 with $t = -1$.

This theorem is also a generalization of results of Vuletić [V2]. In [V2] a 2-parameter generalization of MacMahon's formula related to Macdonald symmetric functions was given and the formula is especially simple in the Hall-Littlewood case. In the Hall-Littlewood case, this is a generating formula for plane partitions weighted by $A_\Pi(t)$. Theorem 7 can be naturally generalized to Macdonald case, but we don't pursue this here.

Let $\text{Cyl}(T, A)$ be the set of all cylindric partitions with period T and profile A . The main result of Section 6 is:

Theorem 8. (*Generalized MacMahon's formula for cylindric partitions; Hall-Littlewood's case*)

$$\sum_{\Pi \in \text{Cyl}(T, A)} A_{\Pi}^{\text{cyl}}(t) q^{|\Pi|} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{nT}} \prod_{\substack{1 \leq i, j \leq T \\ A_i=0, A_j=1}} \frac{1 - tq^{(j-i)_{(T)} + (n-1)T}}{1 - q^{(j-i)_{(T)} + (n-1)T}},$$

where $i_{(T)}$ is the smallest positive integer such that $i \equiv i_{(T)} \pmod{T}$.

The case $t = 0$ is due to Borodin and represents a generating formula for cylindric partitions. Cylindric partitions were introduced and enumerated by Gessel and Krattenthaler [GK]. The result of Borodin could be also proven using Theorem 5 of [GK] and the $\text{SU}(r)$ -extension of Bailey's ${}_6\psi_6$ summation due to Guftavson (equation (7.9) in [GK]) [Kr]. Again Theorem 8 can be naturally generalized to Macdonald case. The trace generating function of those cylindric partitions could also be easily derived from our proof, as done by Okada [O] for the reverse plane partitions case.

The paper is organized as follows. In Section 2 we give a mostly combinatorial proof of the generalized MacMahon formula. In Section 3 we use nonintersecting paths and Gessel–Viennot result's to obtain the hook-length formulas for plane overpartitions for reverse plane partitions of a given shape. In Section 4 we make the connection between tilings and plane overpartitions. In Section 5 we construct a bijection between matrices and pairs of plane overpartitions and obtain a generating formula for plane overpartitions with bounded part size. In Section 6 we give the hook formula for reverse plane partitions contained in a given shape and the 1-parameter generalization of the generating formula for cylindric partitions.

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2. PLANE PARTITIONS AND HALL–LITTLEWOOD FUNCTIONS

In this Section, we give an alternative proof of the generalization of MacMahon formula due to the third author [V2]. Our proof is mostly combinatorial as it uses a bijection between plane partitions and pairs

of strict plane partitions of the same shape and the combinatorial description of the Hall–Littlewood functions [Mac].

Let $\mathcal{P}(r, c)$ be the set of plane partitions with at most r rows and c columns. Given a plane partition Π , let $A_\Pi(t)$ be the polynomial defined in (1.6), as $A_\Pi(t) = \prod_{r \text{ border component}} (1 - t^{\text{level}(r)})$.

Recall that Theorem 2 states that

$$\sum_{\Pi \in \mathcal{P}(r, c)} A_\Pi(t) q^{|\Pi|} = \prod_{i=1}^r \prod_{j=1}^c \frac{1 - tq^{i+j-1}}{1 - q^{i+j-1}}.$$

Any plane partition Π is in bijection with a sequence of partitions $(\pi^{(1)}, \pi^{(2)}, \dots)$. This sequence is such that $\pi^{(i)}$ is the shape of the entries greater than or equal to i in Π .

4433

For example if $\Pi = \begin{smallmatrix} 4433 \\ 3332 \\ 1 \end{smallmatrix}$, the corresponding sequence is $((4, 4, 1), (4, 4), (4, 3), (2))$.

Note that the plane partition Λ is column strict if and only if $\lambda^{(i)} / \lambda^{(i+1)}$ is a horizontal strip.

We use a bijection between pairs of column strict plane partitions (Σ, Λ) and plane partitions Π due to Bender and Knuth [BK]. We suppose that (Σ, Λ) are of the same shape λ and that the corresponding sequences are $(\sigma^{(1)}, \sigma^{(2)}, \dots)$ and $(\lambda^{(1)}, \lambda^{(2)}, \dots)$

Given a plane partition $\Pi = (\Pi_{i,j})$, we define the entries of diagonal x to be the partition $(\Pi_{i,j})$ with $i, j \geq 1$ and $j - i = x$. The bijection is such that the entries of diagonal x of Π are $\sigma^{(x+1)}$ if $x \geq 0$ and $\lambda^{(-x-1)}$ otherwise. Note that as Λ and Σ have the same shape, the entries of the main diagonal ($x = 0$) are $\sigma^{(1)} = \lambda^{(1)}$.

4444

For example, start with $\Sigma = \begin{smallmatrix} 4444 \\ 2221 \\ 111 \end{smallmatrix}$ whose sequence is $((4, 4, 3), (4, 3), (4), (4))$;

4433

and $\Lambda = \begin{smallmatrix} 3322 \\ 3322 \\ 111 \end{smallmatrix}$, whose sequence is $((4, 4, 3), (4, 4), (4, 2), (2))$ and get

$$\Pi = \begin{smallmatrix} 4444 \\ 443 \\ 443 \\ 22 \end{smallmatrix}.$$

This construction implies that :

$$\begin{aligned} |\Pi| &= |\Sigma| + |\Lambda| - |\lambda| ; \\ A_\Pi(t) &= \frac{\varphi_\Sigma(t)\varphi_\Lambda(t)}{b_\lambda(t)}; \end{aligned}$$

with

$$b_\lambda(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t); \quad \phi_r(t) = \prod_{j=1}^r (1 - t^j);$$

and

$$\phi_\Lambda(t) = \prod_{i \geq 1} \phi_{\lambda^{(i)}/\lambda^{(i+1)}}(t);$$

given a horizontal strip $\theta = \lambda/\mu$,

$$\phi_\theta(t) = \prod_{i \in I} (1 - t^{m_i(\lambda)}),$$

where $m_i(\lambda)$ is the multiplicity of i in λ and I is the set of integers such that $\theta'_i = 1$ and $\theta'_{i+1} = 0$. See [Mac], Chapter III Sections 2 and 5.

Indeed

- Each factor $(1 - t^i)$ in $b_\lambda(t)$ is in one-to-one correspondence with a border component of level i that goes through the main diagonal of Π .
- Each factor $(1 - t^i)$ in $\phi_\Sigma(t)$ is in one-to-one correspondence with a border component of level i that ends in a non-negative diagonal.
- Each factor $(1 - t^i)$ in $\phi_\Lambda(t)$ is in one-to-one correspondence with a border component of level i that starts in a non-positive diagonal.

Continuing with our example, we have

$$\phi_\Sigma(t) = (1-t)^2(1-t^2); \quad \phi_\Lambda(t) = (1-t)^3(1-t^2); \quad b_\lambda(t) = (1-t)^2(1-t^2);$$

and

$$A_\Pi(t) = (1-t)^3(1-t^2) = \frac{(1-t)^2(1-t^2)(1-t)^3(1-t^2)}{(1-t)^2(1-t^2)}.$$

We recall the combinatorial definition of the Hall–Littlewood functions as done in the book of Macdonald [Mac]. The Hall–Littlewood function $Q_\lambda(x; t)$ can be defined as

$$Q_\lambda(x; t) = \sum_{\substack{\Lambda \\ \text{sh}(\Lambda) = \lambda}} \phi_\Lambda(t) x^\Lambda;$$

where $x^\Lambda = x_1^{\alpha_1} x_2^{\alpha_2} \dots$ and α_i is the number of entries equal to i in Λ . See [Mac] Chapter III, equation (5.11).

A direct consequence of the preceding bijection is that, the entries of Σ are less or equal to r and the entries of Λ are less or equal to c if and only if Π is in $\mathcal{P}(r, c)$. Therefore :

$$\sum_{\Pi \in \mathcal{P}(r, c)} A_\Pi(t) q^{|\Pi|} = \sum_{\lambda} \frac{Q_\lambda(q, \dots, q^r, 0, \dots; t) Q_\lambda(q^0, \dots, q^{c-1}, 0, \dots; t)}{b_\lambda(t)}.$$

Finally, we need equation (4.4) in Chapter III of [Mac].

$$(2.1) \quad \sum_{\lambda} \frac{Q_\lambda(x; t) Q_\lambda(y; t)}{b_\lambda(t)} = \prod_{i, j} \frac{1 - tx_i y_j}{1 - x_i y_j};$$

With the substitutions $x_i = q^i$ for $1 \leq i \leq r$ and 0 otherwise and $y_j = q^{j-1}$ for $1 \leq j \leq c$ and 0 otherwise, we get the result.

3. NONINTERSECTING PATHS

3.1. Plane overpartitions of a given shape. In this section we represent plane overpartitions as nonintersecting paths and using results from [GV] we prove Theorems 3 and 4. A similar approach was used for example in [Br] to compute Super-Schur functions.

We construct a bijection between the set of paths from $(0, 0)$ to (x, k) and the set of overpartitions with at most k parts less or equal to x . Given an overpartition the corresponding path consists of North and East edges that form the border of the Ferrer diagram of the overpartition except for corners containing an overlined entry where we substitute a pair of North and East edge with an North-East edge. For example, the path corresponding to the overpartition $(5, \bar{5}, 3, 3, \bar{2})$ is shown on Figure 4.

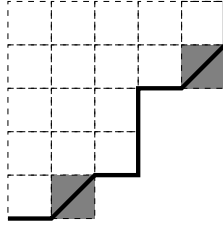


FIGURE 4. Paths and overpartitions

To each overpartition λ we associate a weight equal to $a^{o(\lambda)} q^{|\lambda|}$, where $o(\lambda)$ is the number of overlined parts. To have the same weight on the corresponding path we introduce the following weights on edges. We

assign weight 1 to East edges, q^i to North edges on level i and weight aq^{i+1} to North-East edges joining levels i and $i+1$. For convenience we use these paths and weights to construct a bijection between plane overpartitions and sets of nonintersecting paths.

Lemma 1. [CL] *The generating functions of overpartitions with at most k parts is given by*

$$(3.1) \quad \sum_{l(\lambda) \leq k} a^{o(\lambda)} q^{|\lambda|} = \frac{(-aq)_k}{(q)_k}$$

and exactly k parts is given by

$$(3.2) \quad \sum_{l(\lambda)=k} a^{o(\lambda)} q^{|\lambda|} = q^k \frac{(-a)_k}{(q)_k}.$$

Proof. Let λ be an overpartition seen as the corresponding path. Let $r_i(\lambda)$ be the number of North-East edges joining levels i and $i+1$. Obviously, it takes values 0 or 1. Let $s_i(\lambda)$ be the number of East edges on level i . Then

$$\sum_{l(\lambda) \leq k} a^{o(\lambda)} q^{|\lambda|} = \prod_{i=1}^k \sum_{r_i=0,1} (aq^i)^{r_i} \prod_{i=1}^k \sum_{s_i=0}^{\infty} (q^i)^{s_i} = \frac{(-aq)_k}{(q)_k}.$$

Now, if λ is an overpartition with exactly k parts then $r_k(\lambda)$ or $s_k(\lambda)$ must be nonzero. Then using (3.1) we obtain

$$\sum_{l(\lambda)=k} a^{o(\lambda)} q^{|\lambda|} = q^k \left[\frac{(-aq)_k}{(q)_k} + a \frac{(-aq)_{k-1}}{(q)_{k-1}} \right] = q^k \frac{(-a)_k}{(q)_k}.$$

□

For a plane overpartition Π of shape λ we construct a set of nonintersecting paths using paths from row overpartitions where the starting point of the path corresponding to the i th row is shifted upwards by $\lambda_1 - \lambda_i + i - 1$ so that the starting point is $(0, \lambda_1 - \lambda_i + i - 1)$. That way we obtain a bijection between the set of nonintersecting paths from $(0, \lambda_1 - \lambda_i + i - 1)$ to $(x, \lambda_1 + i - 1)$, where i goes from 1 to $\ell(\lambda)$ and the set of plane overpartitions whose i th row overpartition has at most λ_i and at least λ_{i-1} parts with x greater or equal to the largest part. The weights on the edges correspond to the weight $a^{o(\Pi)} q^{|\Pi|}$. Figure 5 (see also Figure 8) shows the corresponding set of nonintersecting paths for

$x = 8$ and the plane overpartition

$$\begin{array}{ccccccc} 7 & 4 & \bar{3} & 2 & \bar{2} & & \\ 3 & 3 & \bar{3} & \bar{2} & & & \\ \bar{3} & 2 & \bar{2} & & & & \\ 2 & & & & & & \end{array}.$$

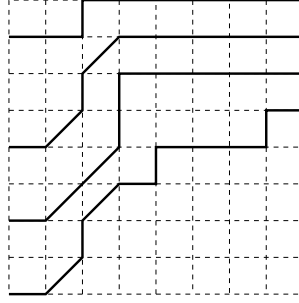


FIGURE 5. Nonintersecting paths

Definition 1. For a partition λ we define $M_\lambda(a; q)$ to be a $\ell(\lambda) \times \ell(\lambda)$ matrix whose (i, j) th entry is given by

$$\frac{(-a)_{\lambda_j+i-j}}{(q)_{\lambda_j+i-j}}.$$

For a partition λ let $\mathcal{B}(\lambda)$ and $\mathcal{S}(\lambda)$ be as in the introduction, i.e. the sets of all plane overpartitions bounded by shape λ and of the shape λ , respectively.

Proposition 1. Let λ be a partition. The weighted generating function of plane overpartitions such that the i^{th} row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts is given by

$$\sum_{\Pi \in \mathcal{B}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = \det M_\lambda(aq; q).$$

Proof. From (3.1) we have that the limit when x goes to infinity of the number of paths from $(0, 0)$ to (x, k) is $(-aq)_k / (q)_k$. Using Corollary 2 of [GV] we have that $\det M_\lambda(aq; q)$ is the limit when x goes to infinity of the generating function of $\ell(\lambda)$ nonintersecting paths going from $(0, \lambda_1 + i - 1 - \lambda_i)$ to $(x, \lambda_1 + i - 1)$. Thanks to the bijection between paths and overpartitions this is also the generating function for overpartitions whose i th row overpartition has at most λ_i and at least λ_{i+1} parts. \square

Proposition 2. *Let λ be a partition. The weighted generating function of plane overpartitions of shape λ is given by*

$$\sum_{\Pi \in \mathcal{S}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{|\lambda|} \det M_\lambda(a; q).$$

Proof. The proof is the same as the proof of Proposition 1. Here we just use (3.2) instead of (3.1). \square

We now prove a lemma that will allow us to give a product formula for these generating functions.

Lemma 2.

$$(3.3) \quad \det M_\lambda(a; q) = q^{\sum_i (i-1)\lambda_i} \prod_{(i,j) \in \lambda} \frac{1 + aq^{c_{i,j}}}{1 - q^{h_{i,j}}}.$$

Proof. We prove the lemma by induction on the number of columns of λ . For the empty partition the lemma is true. So we assume that λ is nonempty and that for all μ having less columns than λ the lemma holds, i.e.

$$(3.4) \quad \det M_\mu(a; q) = q^{\sum_i (i-1)\mu_i} \prod_{(i,j) \in \mu} \frac{1 + aq^{c_{i,j}}}{1 - q^{h_{i,j}}}.$$

Suppose that $\ell(\lambda) = k$. Let $M = M_\lambda(a; q)$ and $Q = M_{\bar{\lambda}}(aq; q)$, where $\bar{\lambda} = (\lambda_1 - 1, \dots, \lambda_k - 1)$. Therefore,

$$Q(i, j) = \frac{(-aq)_{\lambda_j + i - j - 1}}{(q)_{\lambda_j + i - j - 1}}.$$

Our goal is to show that

$$(3.5) \quad \det M = q^{\binom{k}{2}} \prod_{i=1}^k \frac{1 + aq^{c_{i,1}}}{1 - q^{h_{i,1}}} \det Q,$$

because then (3.3) follows by induction.

We introduce two $k \times k$ matrices N and P with the following properties:

$$(3.6) \quad d_{M/N} := \frac{\det M}{\det N} = \prod_{i=1}^k \frac{(-a)_{h_{i,1}}}{(q)_{h_{i,1}}},$$

$$(3.7) \quad d_{N/P} := \frac{\det N}{\det P} = \frac{\prod_{i=0}^{k-1} (a + q^i)}{(1 + a)^k} = q^{\binom{k}{2}} \frac{\prod_{i=1}^k (1 + aq^{c_{i,1}})}{(1 + a)^k},$$

$$(3.8) \quad d_{P/Q} := \frac{\det P}{\det Q} = \prod_{i=1}^k \frac{(q)_{h_{i,1}-1}}{(-aq)_{h_{i,1}-1}}.$$

These properties imply (3.5).

Matrices N and P are given by

$$N(i, j) = M(i, j) \frac{(q)_{h_{j,1}}}{(-a)_{h_{j,1}}} = \frac{(-a)_{\lambda_j+i-j}(q)_{h_{j,1}}}{(q)_{\lambda_j+i-j}(-a)_{h_{j,1}}}$$

and

$$P(i, j) = Q(i, j) \frac{(q)_{h_{j,1}-1}}{(-aq)_{h_{j,1}-1}} = \frac{(-aq)_{\lambda_j+i-j-1}(q)_{h_{j,1}-1}}{(q)_{\lambda_j+i-j-1}(-aq)_{h_{j,1}-1}}.$$

Properties (3.6) and (3.8) follow immediately. We will prove that

$$(3.9) \quad P(i, j) = \frac{(1+a)N(i, j) - (1 - q^{k-i})P(i+1, j)}{a + q^{k-i}}, \quad i < k,$$

which will imply (3.7).

Let

$$S = (1+a)N(i, j) - (1 - q^{k-i})P(i+1, j).$$

Note that $h_{j,1} = \lambda_j - j + k$. Then

$$\begin{aligned} S &= \frac{(1+a)(-a)_{\lambda_j+i-j}(q)_{h_{j,1}}}{(q)_{\lambda_j+i-j}(-a)_{h_{j,1}}} - \frac{(1 - q^{k-i})(-aq)_{\lambda_j+i-j}(q)_{h_{j,1}-1}}{(q)_{\lambda_j+i-j}(-aq)_{h_{j,1}-1}} \\ &= \left[\frac{(1+a)(1 - q^{h_{j,1}})}{1 - q^{\lambda_j+i-j}} - \frac{(1 - q^{k-i})(1 + aq^{\lambda_j+i-j})}{1 - q^{\lambda_j+i-j}} \right] P(i, j) \\ &= (a + q^{k-i})P(i, j) \end{aligned}$$

and thus (3.9) holds. \square

Then using this lemma we obtain the product formulas for the generating functions from Propositions 1 and 2. Those are the hook-length formulas given in Theorems 3 and 4.

Stanley's hook-content formula states the generating function of column strict plane partitions of shape λ where the entries are less or equal to n is

$$\prod_{(i,j) \in \lambda} \frac{1 + q^{n+c_{i,j}}}{1 - q^{h_{i,j}}}.$$

See Theorem 7.21.2 of [Sta]. It can be seen as a special case of the hook-content formula from Theorem 3 for $a = -q^n$. Indeed there is an easy sign reversing involution on plane overpartitions where the overlined entries are larger than n and where the sign of each entry is - if the entry is overlined (and greater than n) and + otherwise and the sign of the overpartition is the product of the signs of the entries. We

define now this involution. If the largest entry of the plane overpartition is greater than n then if it appears overlined in the first row then take off the overline and otherwise overline the rightmost occurrence of the largest entry in the first row. Note that this changes the sign of the plane overpartition. If the largest entry is at most n then do nothing.

Now, we want to obtain a formula for plane overpartitions with at most r rows and c columns

Proposition 3. *The weighted generating function of plane overpartitions with at most r rows and c columns*

$$\sum_{c \geq \lambda_1 \geq \dots \geq \lambda_{(r-1)/2} \geq 0} \det M_{(c, \lambda_1, \lambda_1, \dots, \lambda_{(r-1)/2}, \lambda_{(r-1)/2})}(a; q)$$

if r is odd and

$$\sum_{n \geq \lambda_1 \geq \dots \geq \lambda_{r/2} \geq 0} \det M_{(\lambda_1, \lambda_1, \dots, \lambda_{r/2}, \lambda_{r/2})}(a; q)$$

otherwise.

In particular, the weighted generating function of all overpartitions is:

$$\sum_{\lambda_1 \geq \dots \geq \lambda_k} \det M_{(\lambda_1, \lambda_1, \dots, \lambda_k, \lambda_k)}(a; q)$$

Proof. This is a direct consequence of Proposition 1. \square

We will use this result to get another “symmetric function” proof of the shifted MacMahon formula.

3.2. The shifted MacMahon formula. The shifted MacMahon formula obtained in [FW, V1, V2]:

$$\sum_{\substack{\Pi \text{ is a plane} \\ \text{overpartition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n$$

can be obtained from symmetric functions using the results of the preceding Subsection.

We showed how Stanley’s hook-content formula follows from Theorem 3. Alternatively, we could use Stanley’s hook-content formula and symmetric functions to prove Lemma 2 that further implies Theorems 3 and 4.

A specialization is an algebra homomorphism between the algebra of symmetric functions and \mathbb{C} . We use three standard bases for the algebra of symmetric functions: Schur functions s , complete symmetric

functions h and power sums p (see [Mac]). If ρ is a specialization we denote their images with $s|_\rho$, $h|_\rho$ and $p|_\rho$.

Let $\rho(a) : \Lambda \rightarrow \mathbb{C}[a]$ be an algebra homomorphism given with

$$h_n|_{\rho(a)} = \frac{(-a)_n}{(q)_n}.$$

Since, for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$

$$s_\lambda = \det(h_{\lambda_i - i + j})$$

we have from Definition 1 that

$$(3.10) \quad s_\lambda|_{\rho(a)} = M_\lambda(a; q).$$

Now, we show that

$$p_n|_{\rho(a)} = \frac{1 - (-a)^n}{1 - q^n}.$$

This follows from (2.2.1) of [A]:

$$1 + \sum_{n=1}^{\infty} \frac{(-a)_n}{(q)_n} t^n = \prod_{n=0}^{\infty} \frac{1 + atq^n}{1 - tq^n}$$

and

$$P(t) = \frac{H'(t)}{H(t)},$$

where H and P are generating functions for h_n and p_n .

Recall the well known formula (see for example Theorem 7.21.2 of [Sta]):

$$(3.11) \quad s_\lambda(1, q, \dots, q^n, 0, \dots) = q^{b(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{n+c(x)+1}}{1 - q^{h(x)}}.$$

Observe that $\rho(q^n)$ is a specialization given with the evaluation $x_i = q^i$ for $i = 1, \dots, n$ and 0 otherwise. Then (3.11) says

$$s_\lambda|_{\rho(q^n)} = q^{b(\lambda)} \prod_{x \in \lambda} \frac{1 - q^n q^{c(x)+1}}{1 - q^{h(x)}}, \quad \text{for every } n.$$

This implies that

$$s_\lambda|_{\rho(a)} = q^{b(\lambda)} \prod_{x \in \lambda} \frac{1 + aq^{c(x)+1}}{1 - q^{h(x)}},$$

because we have polynomials in a on both sides and equality is satisfied for infinitely many values ($a = q^n$, for every n). So, this gives an alternate proof of Lemma 2.

The shifted MacMahon formula can be obtained from (3.10) and Proposition 3. We have that

$$\sum_{\substack{\Pi \text{ is a plane} \\ \text{overpartition}}} q^{|\Pi|} = \sum_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k} s_{(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_k, \lambda_k)}|_{\rho_1} = \sum_{\lambda' \text{ even}} s_{\lambda}|_{\rho_1},$$

where λ' is the transpose of λ and a partition is even if it has even parts and $\rho_1 = \rho(1)$.

By Ex. 5 b) on p. 77 of [Mac] we have that

$$\sum_{\lambda' \text{ even}} s_{\lambda} = \prod_{i < j} \frac{1}{x_i - x_j} = \exp \left[\sum_{n=1}^{\infty} \frac{1}{2n} (p_n^2 - p_{2n}) \right].$$

So,

$$\begin{aligned} \sum_{\lambda' \text{ even}} s_{\lambda}|_{\rho_1} &= \exp \left[\sum_{n=1,3,5,\dots} \frac{1}{2n} \left(\left(\frac{1+q^n}{1-q^n} \right)^2 - 1 \right) \right] \\ &= \exp \left[\sum_{n=1,3,5,\dots} \frac{2q^n}{n(1-q^n)^2} \right] \\ &= \exp \left[\sum_{n=1,3,5,\dots} \frac{2}{n} p_n^2|_{\rho_2} \right], \end{aligned}$$

where the specialization ρ_2 is given with

$$p_n|_{\rho_2} = \frac{q^{n/2}}{1-q^n}.$$

This means that the specialization ρ_2 is actually an evaluation given with $x_i = q^{(2i+1)/2}$. Since

$$\exp \left[\sum_{n=1,3,5,\dots} \frac{2}{n} p_n^2 \right] = \prod_{i,j=1}^{\infty} \frac{1+x_i x_j}{1-x_i x_j}$$

we obtain

$$\exp \left[\sum_{n=1,3,5,\dots} \frac{2}{n} p_n^2|_{\rho_2} \right] = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n$$

which proves the shifted MacMahon formula.

3.3. Reverse plane overpartitions. In this Section we construct a bijection between the set of all reverse plane overpartitions and sets of nonintersecting paths whose endpoints are not fixed. We use this bijection and Stembridge's results [Ste] to obtain a Pfaffian formula for the generating function of reverse plane overpartitions of a given shape.

Evaluating the Pfaffian we obtain the hook formula for reverse plane overpartitions due to Okada. Namely, if $\mathcal{S}^R(\lambda)$ is the set of all reverse plane partitions of shape λ then

Theorem 9. *The generating function of reverse plane overpartitions of shape λ is*

$$\sum_{\Pi \in \mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j) \in \lambda} \frac{1 + q^{h_{i,j}}}{1 - q^{h_{i,j}}}.$$

We construct a weight preserving bijection between reverse plane overpartitions and sets of nonintersecting paths on a triangular lattice in a similar fashion as in Section 2. The lattice consists of East, North and North-East edges. East edges have weight 1, North edges on level i have weight q^i and North-East edges joining levels i and $i+1$ have weight q^i . The weight of a set of nonintersecting paths p is the product of the weights of their edges and is denoted with $w(p)$. Let Π be a reverse plane overpartition whose positive entries form a skew shape λ/μ and let $\ell = \ell(\lambda)$. Then Π can be represented by a set of n nonintersecting lattice paths such that

- the departure points are $(0, \mu_i + \ell - i)$ and
- the arrivals points are $(x, \lambda_i + \ell - i)$,

for a large enough x and $i = 1, \dots, \ell$. For example let $x = 8$, $\lambda = (5, 4, 2, 2)$ and $\mu = (2, 1)$. Figure 6 shows the corresponding set of nonintersecting paths for the reverse plane overpartition of shape $\lambda \setminus \mu$

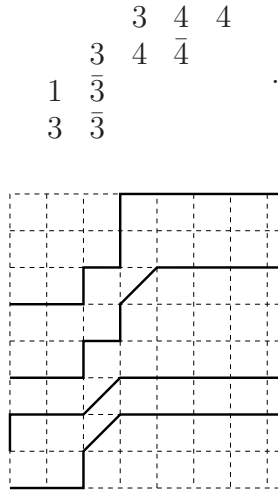


FIGURE 6. Nonintersecting paths and reverse plane overpartitions

This implies that all reverse plane overpartitions of shape λ can be represented by nonintersecting lattice paths such that

- the departure points are a l -element subset of $\{(0, i) \mid i \geq 0\}$ and
- the arrivals points are $(x, \lambda_i + l - i)$,

with $x \rightarrow \infty$.

Now, for $r_1 > r_2 > \dots > r_n \geq 0$ we define

$$W(r_1, r_2, \dots, r_n) = \lim_{x \rightarrow \infty} \sum_{p \in P(x; r_1, \dots, r_n)} w(p),$$

where $P(x; r_1, r_2, \dots, r_n)$ is the set of all nonintersecting paths joining a n -element subset of $\{(0, i) \mid i \geq 0\}$ with $\{(x, r_1), \dots, (x, r_n)\}$. Note that for $r_1 > r_2 > \dots > r_n > 0$

$$(3.12) \quad W(r_1, \dots, r_n, 0) = W(r_1 - 1, \dots, r_n - 1).$$

By Stembridge's Pfaffian formula for the sum of the weights of nonintersecting paths where departure points are not fixed [Ste] we obtain

$$W(r_1, r_2, \dots, r_n) = \text{Pf}(D),$$

where if n is even D is $n \times n$ skew-symmetric matrix defined by $D_{i,j} = W(r_i, r_j)$ for $1 \leq i < j \leq n$ and if n is odd D is $(n+1) \times (n+1)$ skew-symmetric matrix defined by $D_{i,j} = W(r_i, r_j)$ and $D_{i,n+1} = W(r_i)$ for $1 \leq i < j < n+1$.

Lemma 3. *Let $r > s \geq 0$. Then*

$$(3.13) \quad W(s) = \frac{(-q)_s}{(q)_s},$$

$$(3.14) \quad W(r, s) = \frac{(-q)_r}{(q)_r} \cdot \frac{(-q)_s}{(q)_s} \cdot \frac{1 - q^{r-s}}{1 + q^{r-s}}.$$

Proof. From Lemma 1 we have that the generating function of overpartitions with at most n parts is $M(n) = (-q)_n / (q)_n$ and with exactly n parts is $P(n) = q^n (-1)_n / (q)_n$. We set $M(n) = P(n) = 0$, for negative n .

From this lemma it follows:

$$(3.15) \quad \sum_{i=0}^n P(i) = M(n).$$

Thus,

$$W(s) = M(s) = \frac{(-q)_s}{(q)_s} \quad \text{and} \quad W(r, 0) = M(r-1) = \frac{(-q)_{r-1}}{(q)_{r-1}}.$$

We prove (3.14) by induction on s . The formula for the base case $s = 0$ holds by above. So, we assume $s \geq 1$.

By Gessel-Viennot's determinantal formula [GV] we have that

$$W(r, s) = \sum_{i=0}^s \sum_{j=i+1}^r P(r-j)P(s-i) - P(r-i)P(s-j).$$

Summing over j and using (3.15) we obtain

$$W(r, s) = \sum_{i=0}^s P(s-i)M(r-i-1) - P(r-i)M(s-1-i).$$

Then

$$W(r, s) = W(r-1, s-1) + P(s)M(r-1) - P(r)M(s-1).$$

It is enough to prove that

$$(3.16) \quad \frac{P(s)M(r-1) - P(r)M(s-1)}{W(r-1, s-1)} = \frac{2(q^r + q^s)}{(1-q^r)(1-q^s)}$$

and (3.14) follows by induction. Now,

$$\begin{aligned} & P(s)M(r-1) - P(r)M(s-1) = \\ &= q^s \frac{(-1)_s}{(q)_s} \cdot \frac{(-q)_{r-1}}{(q)_{r-1}} - q^r \frac{(-1)_r}{(q)_r} \cdot \frac{(-q)_{s-1}}{(q)_{s-1}} \\ &= \frac{(-q)_{r-1}}{(q)_{r-1}} \cdot \frac{(-q)_{s-1}}{(q)_{s-1}} \cdot \frac{1+q^{r-s}}{1-q^{r-s}} \cdot \frac{2(q^r + q^s)}{(1-q^r)(1-q^s)}. \end{aligned}$$

Using inductive hypothesis for $W(r-1, s-1)$ we obtain (3.16). \square

Let F_λ be the generating function of reverse plane overpartitions of shape λ . Then using the bijection we have constructed we obtain that

$$F_\lambda = W(\lambda_1 + l - 1, \lambda_2 + l - 2, \dots, \lambda_l)$$

which, after applying Stembridge's result, gives us the Pfaffian formula. This Pfaffian formula can be expressed as a product after the following observations.

Let M be $2k \times 2k$ skew-symmetric matrix. One of definitions of the Pfaffian is the following:

$$\text{Pf}(M) = \sum_{\Pi=(i_1, j_1) \dots (i_n, j_n)} \text{sgn}(\Pi) M_{i_1, j_1} M_{i_2, j_2} \cdots M_{i_n, j_n},$$

where the sum is over all of perfect matchings (or fixed point free involutions) of $[2n]$. Also, for $r > s > 0$

$$\begin{aligned} W(s) &= \frac{1+q^s}{1-q^s} W(s-1) \\ W(r, s) &= \frac{1+q^r}{1-q^r} \cdot \frac{1+q^s}{1-q^s} W(r-1, s-1) \end{aligned}$$

Then

$$F_\lambda = \prod_{j=1}^l \frac{1+q^{h_{j,1}}}{1-q^{h_{j,1}}} \cdot W(\lambda_1 + l - 2, \lambda_2 + l - 3, \dots, \lambda_l - 1) = F_{\bar{\lambda}},$$

where $\bar{\lambda} = (\lambda_1 - 1, \dots, \lambda_l - 1)$ if $\lambda_l > 1$ and $\bar{\lambda} = (\lambda_1 - 1, \dots, \lambda_{l-1} - 1)$ if $\lambda_l = 1$ (see (3.12) in this case). Inductively we obtain Theorem 9.

4. DOMINO TILINGS

In [V1] a measure on strict plane partitions was studied. Strict plane partitions are plane partitions where all diagonals are strict partitions, i.e. strictly decreasing sequences. They can also be seen as plane overpartitions where all overlines are deleted. There are $2^{k(\Pi)}$ different plane overpartitions corresponding to a same strict plane partition Π .

Alternatively, a strict plane partition can be seen as a subset of $\mathbb{N} \times \mathbb{Z}$ consisting of points (t, x) where x is a part of the diagonal partition indexed by t , see Figure 7. We call this set the 2-dimensional diagram of that strict plane partition. The connected components are connected sets (no holes) on the same horizontal line. The 2-dimensional diagram of a plane overpartition is the 2-dimensional diagram of its corresponding strict plane partition.

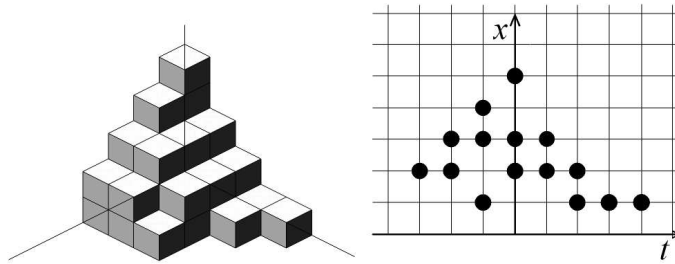


FIGURE 7. A 2-dimensional diagram

The measure studied in [V1] assigns to each strict plane partition a weight equal to $2^{k(\Pi)} q^{|\Pi|}$. The limit shape of this measure is given in terms of the Ronkin function of the polynomial $P(z, w) = -1 +$

$z + w + zw$ and it is parameterized on the domain representing half of the amoeba of this polynomial. This polynomial is also related to plane tilings with dominoes. This, as well as some other features like similarities in correlation kernels, see [J, V1], suggested that the connection between this measure and domino tilings is likely to exist.

Alternatively, one can see this measure as a uniform measure on plane overpartitions, i.e. each plane overpartition Π has a probability proportional to $q^{|\Pi|}$. In Section 3 we have constructed a bijection between overpartitions and nonintersecting paths passing through the elements of the corresponding 2-dimensional diagrams, see Figure 8. The paths consist of edges of three different kinds: horizontal (joining (t, x) and $(t + 1, x)$), vertical (joining $(t, x + 1)$ and (t, x)) and diagonal (joining $(t, x + 1)$ and $(t + 1, x)$). There is a standard way to construct a tiling with dominoes using these paths, see for example [J]. We explain the process below.

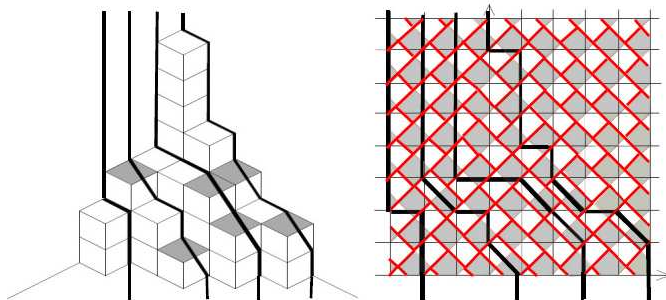


FIGURE 8. A strict plane partition and the domino tiling

We start from \mathbb{R}^2 and color it in a chessboard fashion such that $(1/4, 1/4)$, $(-1/4, 3/4)$, $(1/4, 5/4)$ and $(3/4, 3/4)$ are vertices of a white square. So, the axes of this infinite chessboard form angles of 45 and 135 degrees with axes of \mathbb{R}^2 . A domino placed on this infinite chessboard can be one of the four types: $(1, 1)$, $(-1, -1)$, $(-1, 1)$ or $(1, -1)$, where we say that a domino is of type (x, y) if (x, y) is a vector parallel to the vector whose starting, respectively end point is the middle of the white, respectively black square of that domino.

Now, take a plane overpartition and represent it by a set of non-intersecting paths passing through the elements of the 2-dimensional diagram of this plane overpartition. We cover each edge by a domino that satisfies that the endpoints of that edge are middle points of sides of the black and white square of that domino. That way we obtain a tiling of a part of the plane with dominoes of three types: $(1, 1)$, $(-1, -1)$ and $(1, -1)$. More precisely, horizontal edges correspond to

$(1, 1)$ dominoes, vertical to $(-1, -1)$ and diagonal to $(1, -1)$. To tile the whole plane we fill the rest of it by dominoes of the forth, $(-1, 1)$ type. See Figure 8 for an illustration.

This way we have established a correspondence between plane overpartitions and plane tilings with dominoes. We now give some of the properties of this correspondence. First, we describe how a tiling changes in the case when we add or remove an overline or we add or remove a box from a plane overpartition. We require that when we add/remove an overline or a box we obtain a plane overpartition again. It is enough to consider only one operation adding or removing since they are inverse of each other. In terms of the 2-dimensional diagram of a plane overpartition, adding an overline can occur at all places where (t, x) is in the diagram and $(t + 1, x)$ is not. Removing a box can occur at all places where (t, x) is the diagram and $(t, x - 1)$ is not.

Adding an overline means that a pair of horizontal and vertical edges is replaced by one diagonal edge. This means that the new tiling differs from the old one by replacing a pair of $(1, 1)$ and $(-1, -1)$ dominoes by a pair of $(1, -1)$ and $(-1, 1)$ dominoes, see Figure 9.

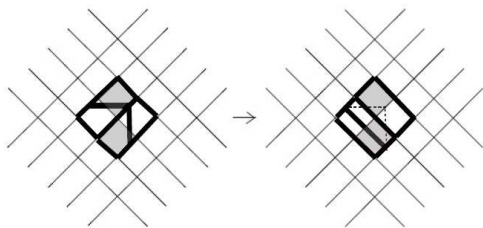


FIGURE 9. Adding an overline

Removing a box from a plane overpartition corresponds to a similar thing. Observe that if a box can be removed then the corresponding part is overlined or it can be overlined to obtain a plane overpartition again. So, it is enough to consider how the tiling changes when we remove a box from an overlined part since we have already considered the case of adding or removing an overline. If we remove a box from an overlined part we change a diagonal edge by a pair of vertical and horizontal edges. This means that the new tiling differs from the old one by replacing a pair of $(1, -1)$ and $(-1, 1)$ dominoes by a pair of $(1, 1)$ and $(-1, -1)$ dominoes, see Figure 10. The difference between removing an overline and removing a box is just in the position where the flipping of dominoes occurs.

We conclude this section by an observation that plane overpartitions of a given shape λ and whose parts are bounded by n are in bijection

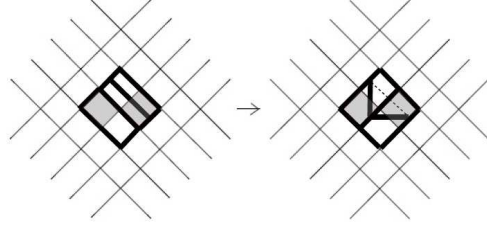


FIGURE 10. Removing a box

with domino tilings of a rectangle $[-\ell(\lambda) + 1, \lambda_1] \times [0, n]$ with certain boundary conditions. These conditions are imposed by the fact that outside of this rectangle nonintersecting paths are just straight lines. We describe the boundary conditions precisely in the proposition below.

Proposition 4. *The set $\mathcal{S}(\lambda) \cup \mathcal{L}(n)$, of all plane overpartitions of shape λ and whose largest part is at most n , is in a bijection with plane tilings with dominoes where a point $(t, x) \in \mathbb{Z} \times \mathbb{R}$ is covered by a domino of type $(-1, -1)$ if*

- $t \leq -\ell(\lambda)$,
- $-\ell(\lambda) < t \leq 0$ and $x \geq n$,
- $t = \lambda_i - i + 1$ for some i and $x \leq 0$,

and a domino of type $(-1, 1)$ if

- $t > \lambda_1$,
- $0 < t \leq \lambda_1$ and $x \geq n + 1/2$,
- $t \neq \lambda_i - i + 1$ for all i and $x \leq -1/2$.

For the example from Figure 8 the boundary conditions are shown in Figure 11.

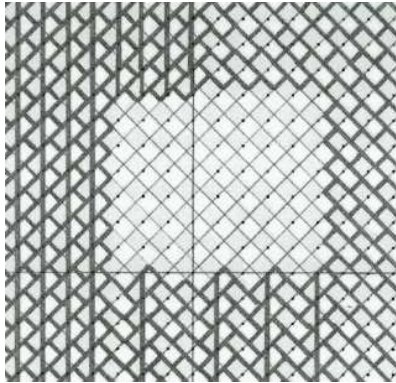


FIGURE 11. Boundary conditions

5. ROBINSON-SCHENSTED-KNUTH (RSK) TYPE ALGORITHM FOR PLANE OVERPARTITIONS

In this section we are going to give a bijection between matrices and pairs of plane overpartitions of the same shape. This bijection is inspired by the algorithm RS2 of Berele and Remmel [BeR] which gives a bijection between matrices and pairs of (k, ℓ) -semistandard tableaux. It is shown in [Sav] that there is a bijection between (n, n) -semistandard tableaux and plane overpartitions whose largest entry is at most n . This bijection is based on the jeu de taquin.

We then apply properties of this algorithm to enumerate plane overpartitions, as done by Bender and Knuth [BK] for plane partitions.

5.1. The RSK algorithm. We start by describing a bijection between matrices and pairs of numbers. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a $2n \times 2n$ matrix, made of four $n \times n$ blocks A, B, C and D . The blocks A and D are non-negative integer matrices, and B and C are $\{0, 1\}$ matrices. The encoding of M into pairs is made using the following rules:

- for each non-zero entry $a_{i,j}$ of A , we create $a_{i,j}$ pairs $\begin{pmatrix} i \\ j \end{pmatrix}$,
- for each non-zero entry $b_{i,j}$ of B , we create one pair $\begin{pmatrix} i \\ \bar{j} \end{pmatrix}$,
- for each non-zero entry $c_{i,j}$ of C , we create one pair $\begin{pmatrix} \bar{i} \\ j \end{pmatrix}$,
- for each non-zero entry $d_{i,j}$ of D , we create $d_{i,j}$ pairs $\begin{pmatrix} \bar{i} \\ \bar{j} \end{pmatrix}$.

It is clear that this encoding defines a one-to-one correspondence between matrices and pairs of numbers.

Example 1. Let $M = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. After encoding M , we obtain

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} \\ 1 \end{pmatrix}, \begin{pmatrix} \bar{1} \\ 2 \end{pmatrix}, \begin{pmatrix} \bar{1} \\ \bar{2} \end{pmatrix}, \begin{pmatrix} \bar{2} \\ \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{2} \\ \bar{2} \end{pmatrix}.$$

From now, we fix the order $\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 < \dots$, we sort the pairs to create a two-line array L such that :

- the first line is a non-increasing sequence
- if two entries of the first line are equal and overlined (resp. non-overlined) then the corresponding entries in the second line are in weakly increasing (resp. decreasing) order.

Example 2 (example 1 continued). *After sorting, we obtain the two-line array L*

$$\begin{pmatrix} 2, 2, 2, \bar{2}, \bar{2}, 1, 1, 1, \bar{1}, \bar{1}, \bar{1} \\ 1, 1, \bar{1}, \bar{1}, \bar{2}, 2, 2, \bar{1}, 1, \bar{2}, 2 \end{pmatrix}$$

We now describe the part of the bijection which is the *insertion algorithm*. It is based on an algorithm proposed by Knuth in [K] and quite similar to the algorithm RS2 of [BeR].

We first define how to insert the entry j into a plane overpartition P .

- (1) $i \leftarrow 1$
- (2) $x_i \leftarrow j$
- (3) If x_i is smaller than all entries of the i^{th} row of P , then insert x_i at the end of this row.
- (4) Otherwise let j_i the index such that $P_{i,j_i-1} \geq x_i > P_{i,j_i}$.
 - (a) If $P_{i,j_i-1} = x_i$ and P_{i,j_i-1} is overlined, then $x_{i+1} \leftarrow x_i$.
 - (b) Otherwise
 - (i) $x_{i+1} \leftarrow P_{i,j_i}$
 - (ii) $P_{i,j_i} \leftarrow x_i$
- (5) $i \leftarrow i + 1$. Go to step 3.

Now we define how to insert a pair $\begin{pmatrix} i \\ j \end{pmatrix}$ into a pair of plane overpartitions of the same shape (P, Q) .

- (1) Insert j in P .
- (2) If the insertion ends in column c and row r of P , then insert i in column c and row r in Q .

Now to go from the two-line array L to pairs of plane overpartitions of the same shape goes as follows : start with two empty plane overpartitions and insert each pair of L going from left to right. This is identical to the classical RSK algorithm [K].

Continuing the previous example, we get

$$P = \begin{array}{ccc} 2 & 2 & 2 \\ \bar{2} & 1 & 1 \\ \bar{2} & \bar{1} & \\ 1 & & \\ \bar{1} & & \\ \bar{1} & & \end{array}; \quad Q = \begin{array}{ccc} 2 & 2 & 2 \\ \bar{2} & 1 & 1 \\ \bar{2} & \bar{1} & \\ 1 & & \\ \bar{1} & & \\ \bar{1} & & \end{array}$$

Theorem 10. *There is a one-to-one correspondance between matrices M and pairs of plane overpartitions of the same shape (P, Q) . In this correspondance :*

- k appears in P exactly $\sum_i a_{ik} + c_{ik}$
- \bar{k} appears in P exactly $\sum_i b_{ik} + d_{ik}$
- k appears in Q exactly $\sum_i a_{ik} + b_{ik}$
- \bar{k} appears in Q exactly $\sum_i c_{ik} + d_{ik}$

Proof. The proof is really identical the proof of the RSK algorithm [K] or the RS2 algorithm [BeR]. Details are given in [Sav]. \square

As in [K], we can show that :

Theorem 11. *If insertion algorithm produces (P, Q) with input matrix M , then insertion algorithm produces (Q, P) with input matrix M^t .*

Proof. The proof is again analogous to [K]. Given a two-line array $\begin{pmatrix} u_1, \dots, u_N \\ v_1, \dots, v_N \end{pmatrix}$, we partition the pairs (u_ℓ, v_ℓ) such that (u_k, v_k) and (u_m, v_m) with are in the same class if and only if :

- $u_k \leq u_m$ and if $u_k = u_m$, then u_k is overlined AND
- $v_k \geq v_m$ and if $v_k = v_m$, then v_k is overlined.

Then one can sort each class so that the first entry of each pair appear in non-increasing order and then sort the classes so that the first entries of the first pair of each class are in non-increasing order. For example if the two-line array is

$$\begin{pmatrix} 2, 2, 2, \bar{2}, \bar{2}, 1, 1, 1, \bar{1}, \bar{1}, \bar{1} \\ 1, 1, \bar{1}, \bar{1}, \bar{2}, 2, 2, \bar{1}, 1, \bar{2}, 2 \end{pmatrix},$$

we get the classes

$$C_1 = \{(\bar{2}, \bar{1}), (\bar{2}, 1), (1, \bar{2}), (\bar{1}, \bar{2})\}, \quad C_2 = \{(1, 1)\}, \quad C_3 = \{(1, \bar{1}), (\bar{1}, \bar{1}), (\bar{1}, 1)\}.$$

If the classes are C_1, \dots, C_d with

$$C_i = \{(u_{i1}, v_{i1}), \dots, (u_{in_i}, v_{in_i})\}$$

then the first row of P is

$$v_{1n_1}, \dots, v_{dn_d}$$

and the first row of Q is

$$u_{11}, \dots, u_{dd}.$$

Moreover one constructs the rest of P and Q using the pairs :

$$\cup_{i=1}^d \cup_{j=1}^{n_i-1} \begin{pmatrix} u_{i,j+1} \\ v_{ij} \end{pmatrix}.$$

See Lemma 1 of [K] and [Sav] for a complete proof. As the two-line array corresponding to M^T is obtained by interchanging the two lines of the array and rearranging the columns, the Theorem follows. \square

This implies that $M = M^T$ if and only if $P = Q$. Therefore :

Theorem 12. *There is a one-to-one correspondance between symmetric matrices M and plane overpartitions P . In this correspondance :*

- k appears in P exactly $\sum_i a_{ik} + c_{ik}$
- \bar{k} appears in P exactly $\sum_i b_{ik} + d_{ik}$

5.2. Enumeration of plane overpartitions. From the Theorem 12, we can get directly the generating function for plane overpartitions whose largest entry is at most n . These objects are in bijection with symmetric matrix M of size $2n \times 2n$ with blocks A, B, C and D , each of size $n \times n$. The weight of the plane overpartition is $\sum_{i,j} i(a_{i,j} + b_{i,j} + c_{i,j} + d_{i,j})$. As M is symmetric, the weight of the plane overpartition can also be written as $\sum_{i,j} (i+j)b_{i,j} + \sum_i i(a_{i,i} + d_{i,i}) + \sum_{j < i} (i+j)(a_{i,j} + d_{i,j})$.

Let $O_n(m, k)$ be the number of plane overpartitions of m with k overlined parts and entries at most n and $\mathcal{O}_n(q, a) = \sum_{m,k} O_n(m, k) q^m a^k$. Then

Theorem 13. *The generating function $\mathcal{O}_n(q, a)$ for plane overpartitions whose largest entry is at most n is*

$$\mathcal{O}_n(q, a) = \frac{\prod_{i,j=1}^n (1 + aq^{i+j})}{\prod_{i=1}^n \prod_{j=0}^{i-1} (1 - q^{i+j})(1 - aq^{(i+j)})}.$$

Let $\mathcal{O}(q, a)$ be the limit of $\mathcal{O}_n(q, a)$ when n goes to infinity. A direct consequence of the Theorem is the following.

Corollary 1. *The generating function for plane overpartitions $\mathcal{O}(q, a)$ is*

$$\prod_{i=1}^{\infty} \frac{(1 + aq^i)^{i-1}}{(1 - q^i)^{\lceil i/2 \rceil} (1 - aq^i)^{\lceil i/2 \rceil}}.$$

We can also get some more general result, as in [BK] :

Theorem 14. *The generating function of plane overpartitions whose parts lie in a set S of positive integers is :*

$$(5.1) \quad \prod_{i \in S} \left(\frac{\prod_{j \in S} (1 + aq^{i+j})}{(1 - q^i)(1 - aq^i)} \prod_{\substack{j \in S \\ j < i}} \frac{1}{(1 - q^{i+j})(1 - aq^{i+j})} \right).$$

For example

Corollary 2. *The generating function for plane overpartitions into odd parts is*

$$\prod_{i=1}^{\infty} \frac{(1 + aq^{2i})^{i-1}}{(1 - q^{2i-1})(1 - aq^{2i-1})(1 - q^{2i})^{\lfloor i/2 \rfloor} (1 - aq^{2i})^{\lfloor i/2 \rfloor}}.$$

6. INTERLACING SEQUENCES AND CYLINDRIC PARTITIONS

We want to combine results of [Bo] and [V2] to obtain a 1-parameter generalization of the formula for the generating function of cylindric partitions related to the Hall-Littlewood symmetric functions.

We use definitions of interlacing sequences, profiles, cylindric partitions, polynomials $A_{\Pi}(t)$ and $A_{\Pi}^{\text{cyl}}(t)$ given in Introduction.

For an ordinary partition λ we define a polynomial

$$(6.1) \quad b_{\lambda}(t) = \prod_{i \geq 1} \varphi_{m_i(\lambda)}(t),$$

where $m_i(\lambda)$ denotes the number of times i occurs as a part of λ and $\varphi_r(t) = (1 - t)(1 - t^2) \cdots (1 - t^r)$.

For a horizontal strip $\theta = \lambda/\mu$ we define

$$\begin{aligned} I_{\lambda/\mu} &= \{i \geq 1 \mid \theta'_i = 1 \text{ and } \theta'_{i+1} = 0\} \\ J_{\lambda/\mu} &= \{j \geq 1 \mid \theta'_j = 0 \text{ and } \theta'_{j+1} = 1\} \end{aligned}$$

Let

$$(6.2) \quad \varphi_{\lambda/\mu}(t) = \prod_{i \in I_{\lambda/\mu}} (1 - t^{m_i(\lambda)}) \text{ and } \psi_{\lambda/\mu}(t) = \prod_{j \in J_{\lambda/\mu}} (1 - t^{m_j(\mu)}).$$

Then

$$\varphi_{\lambda/\mu}/\psi_{\lambda/\mu} = b_{\lambda}(t)/b_{\mu}(t).$$

For an interlacing sequence $\Lambda = (\lambda^1, \dots, \lambda^T)$ with profile $A = (A_1, \dots, A_{T-1})$ we define $\Phi_{\Lambda}(t)$:

$$(6.3) \quad \Phi_{\Lambda}(t) = \prod_{i=1}^{T-1} \phi_i(t),$$

where

$$\phi_i(t) = \begin{cases} \varphi_{\lambda^{i+1}/\lambda^i}(t), & A_i = 0, \\ \psi_{\lambda^i/\lambda^{i+1}}(t), & A_i = 1. \end{cases}$$

For $\Lambda = (\lambda^1, \dots, \lambda^T)$ and $M = (\mu^1, \dots, \mu^S)$ such that $\lambda^T = \mu^1$ we define

$$\Lambda \cdot M = (\lambda^1, \dots, \lambda^T, \mu^2, \dots, \mu^S)$$

and

$$\Lambda \cap M = \lambda^T.$$

Then

$$(6.4) \quad A_{\Lambda \cdot M} = \frac{A_\Lambda A_M}{b_{\Lambda \cap M}} \quad \text{and} \quad \Phi_{\Lambda \cdot M} = \Phi_\Lambda \Phi_M.$$

For an interlacing sequence $\Lambda = (\lambda^1, \dots, \lambda^T)$ with profile $A = (A_1, \dots, A_{T-1})$ we define the reverse $\bar{\Lambda} = (\lambda^T, \dots, \lambda^1)$ whose profile is $\bar{A} = (1 - A_{T-1}, \dots, 1 - A_1)$. Then

$$(6.5) \quad A_{\bar{\Lambda}} = A_\Lambda \quad \text{and} \quad \Phi_{\bar{\Lambda}} = \frac{b_{\lambda^1} \Phi_\Lambda}{b_{\lambda^T}}.$$

For an ordinary partition λ we construct an interlacing sequence $\langle \lambda \rangle = (\emptyset, \lambda^1, \dots, \lambda^L)$ of length $L + 1 = l(\lambda) + 1$, where λ^i is obtained from λ by truncating the last $L - i$ parts. Then

$$(6.6) \quad A_{\langle \lambda \rangle} = \Phi_{\langle \lambda \rangle} = b_\lambda.$$

In our earlier paper [V2] (Propositions 2.4 and 2.6) we have shown that for a plane partition Π

$$(6.7) \quad \Phi_\Pi = A_\Pi.$$

Now, more generally

Proposition 5. *If $\Lambda = (\lambda^1, \dots, \lambda^T)$ is an interlacing sequence then*

$$\Phi_\Lambda = \frac{A_\Lambda}{b_{\lambda^1}}.$$

Proof. If we show that the statement is true for sequences with constant profiles then inductively using (6.4) we can show it is true for sequences with arbitrary profile. It is enough to show that the statement is true for sequences with $(0, \dots, 0)$ profile because of (6.5). So, let $\Lambda = (\lambda^1, \dots, \lambda^T)$ be a sequence with $(0, \dots, 0)$ profile. Then $\Pi = \langle \lambda^1 \rangle \cdot \Lambda \cdot \langle \lambda^T \rangle$ is a plane partition and from (6.4), (6.5), (6.6) and (6.7) we obtain that $A_\Pi = A_\Lambda$ and $\Phi_\Pi = b_{\lambda^1} \Phi_\Lambda$. \square

For skew plane partitions and cylindric partitions we obtain the following two corollaries.

Corollary 3. *For a skew plane partition Π we have that $\Phi_\Pi = A_\Pi$.*

Corollary 4. *If Π is a cylindric partition given with $\Lambda = (\lambda^0, \dots, \lambda^T)$ then $\Phi_\Lambda = A_\Pi^{\text{cyl}}$.*

The last corollary comes from the observation that if a cylindric partition Π is given by a sequence $\Lambda = (\lambda^0, \dots, \lambda^T)$ then

$$(6.8) \quad A_\Pi^{\text{cyl}}(t) = A_\Lambda(t)/b_{\lambda^0}(t).$$

In the rest of this section we prove generalized MacMahon's formulas for skew plane partitions and cylindric partitions that are stated in Theorems 7 and 8. The proofs of these theorems were inspired by [Bo], [OR] and [V2]. We use a special class of symmetric functions called Hall-Littlewood functions.

6.1. The weight functions. In this subsection we introduce weights on sequences of ordinary partitions. For that we use Hall-Littlewood symmetric functions P and Q . We recall some of the facts about these functions, but for more details see Chapters III and VI of [Mac]. We follow the notation used there.

Recall that Hall-Littlewood symmetric functions $P_{\lambda/\mu}(x; t)$ and $Q_{\lambda/\mu}(x; t)$ depend on a parameter t and are indexed by pairs of ordinary partitions λ and μ . In the case when $t = 0$ they are equal to ordinary Schur functions and in the case when $t = -1$ to Schur P and Q functions.

The relationship between P and Q functions is given by (see (5.4) of [Mac, Chapter III])

$$(6.9) \quad Q_{\lambda/\mu} = \frac{b_\lambda}{b_\mu} P_{\lambda/\mu},$$

where b is given with (6.1). Recall that (by (5.3) of [Mac, Chapter III] and (6.9))

$$(6.10) \quad P_{\lambda/\mu} = Q_{\lambda/\mu} = 0 \quad \text{unless } \lambda \supset \mu.$$

We set $P_\lambda = P_{\lambda/\emptyset}$ and $Q_\lambda = Q_{\lambda/\emptyset}$. Recall that ((4.4) of [Mac, Chapter III])

$$H(x, y; t) := \sum_{\lambda} Q_{\lambda}(x; t) P_{\lambda}(y; t) = \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j}.$$

A specialization of an algebra \mathcal{A} is an algebra homomorphism $\rho : \mathcal{A} \rightarrow \mathbb{C}$. If ρ and σ are specializations of the algebra of symmetric functions then we write $P_{\lambda/\mu}(\rho; t)$, $Q_{\lambda/\mu}(\rho; t)$ and $H(\rho, \sigma; t)$ for the images of $P_{\lambda/\mu}(x; t)$, $Q_{\lambda/\mu}(x; t)$ and $H(x, y; t)$ under ρ , respectively $\rho \otimes \sigma$. Every map $\rho : (x_1, x_2, \dots) \rightarrow (a_1, a_2, \dots)$ where $a_i \in \mathbb{C}$ and only

finitely many a_i 's are nonzero defines a specialization. These specializations are called evaluations. A multiplication of a specialization ρ by a scalar $a \in \mathbb{C}$ is defined by its images on power sums:

$$p_n(a \cdot \rho) = a^n p_n(\rho).$$

If ρ is a specialization of Λ where $x_1 = a$, $x_2 = x_3 = \dots = 0$ then by (5.14) and (5.14') of [Mac, Chapter VI]

$$Q_{\lambda/\mu}(\rho; t) = \begin{cases} \varphi_{\lambda/\mu}(t) a^{|\lambda| - |\mu|} & \lambda \supset \mu, \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$P_{\lambda/\mu}(\rho; t) = \begin{cases} \psi_{\lambda/\mu}(t) a^{|\lambda| - |\mu|} & \lambda \supset \mu, \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $T \geq 2$ be an integer and $\rho^\pm = (\rho_1^\pm, \dots, \rho_{T-1}^\pm)$ be finite sequences of specializations. For a sequence of partitions $\Lambda = (\lambda^1, \dots, \lambda^T)$ we set the weight function $W(\Lambda; t)$ to be

$$W(\Lambda) = q^{T|\lambda^T|} \sum_M \prod_{n=1}^{T-1} P_{\lambda^n/\mu^n}(\rho_n^-; t) Q_{\lambda^{n+1}/\mu^n}(\rho_n^+; t),$$

where q and t are parameters and the sum ranges over all sequences of partitions $M = (\mu^1, \dots, \mu^{T-1})$.

We can also define the weights using another set of specializations $R^\pm = (R_1^\pm, \dots, R_{T-1}^\pm)$ were $R_i^\pm = q^{\pm i} \rho_i^\pm$. Then

$$W(\Lambda) = \sum_M W(\Lambda, M, R^-, R^+),$$

where

$$W(\Lambda, M, R^-, R^+) = q^{|\Lambda|} \prod_{n=1}^{T-1} P_{\lambda^n/\mu^n}(R_n^-; t) Q_{\lambda^{n+1}/\mu^n}(R_n^+; t).$$

We will focus on two special sums

$$Z_{\text{skew}} = \sum_{\Lambda=(\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)} W(\Lambda)$$

and

$$Z_{\text{cyl}} = \sum_{\substack{\Lambda=(\lambda^0, \lambda^1, \dots, \lambda^T) \\ \lambda^0 = \lambda^T}} W(\Lambda)$$

6.2. Specializations. We show that for a suitably chosen specializations the weight function vanishes for every sequence of ordinary partitions unless this sequence represents an interlacing sequence, in which case it becomes (6.3).

Let $A^- = (A_1^-, \dots, A_{T-1}^-)$ and $A^+ = (A_1^+, \dots, A_{T-1}^+)$ be sequences of 0's and 1's such that $A_k^- + A_k^+ = 1$.

If specializations R^\pm are evaluations given by

$$(6.13) \quad R_k^\pm : x_1 = A_k^\pm, x_2 = x_3 = \dots = 0$$

then $W(\Lambda)$ vanishes unless Λ is an interlacing sequence of profile A^- and in that case

$$W(\Lambda; t) = \Phi_\Lambda(t) q^{|\Lambda|}.$$

Then from Corollaries 3 and 4 we have that

$$Z_{\text{skew}} = \sum_{\Pi \in \text{Skew}(T, A)} A_\Pi(t) q^{|\Pi|}$$

and

$$Z_{\text{cyl}} = \sum_{\Pi \in \text{Cyl}(T, A)} A_\Pi^{\text{cyl}}(t) q^{|\Pi|}$$

where A^- in both formulas is given by the fixed profile A of skew plane partitions, respectively cylindric partitions.

6.3. Partition functions. If ρ^+ is $x_1 = s, x_2 = x_3 = \dots = 0$ and ρ^- is $x_1 = r, x_2 = x_3 = \dots = 0$ then

$$H(\rho^+, \rho^-) = \frac{1 + tsr}{1 - sr}.$$

Thus, for specializations given by (6.13)

$$(6.14) \quad H(\rho_i^+, \rho_j^-) = \frac{1 + tq^{j-i} A_i^+ A_j^-}{1 - q^{j-i} A_i^+ A_j^-}.$$

We have shown in our earlier paper (Proposition 2.2 of [V2]) that

Proposition 6.

$$Z_{\text{skew}}(\rho^-, \rho^+) = \prod_{0 \leq i < j \leq T} H(\rho_i^+, \rho_j^-).$$

Then this proposition together with (6.14) implies Theorem 7. The generating formula for skew plane partitions can also be seen as the generating formula for reverse plane partitions as explained in Introduction.

Each skew plane partition can be represented as an infinite sequence of ordinary partitions by adding infinitely many empty partitions to

the left and right side. That way the profiles become infinite sequences of 0's and 1's. Theorem 7 also gives the generating formula for skew plane partitions of infinite profiles $A = (\dots, A_{-1}, A_0, A_1, \dots)$:

$$(6.15) \quad \sum_{\Pi \in \text{Skew}(A)} A_{\Pi}(t) q^{|\Pi|} = \prod_{\substack{i < j \\ A_i=0, A_j=1}} \frac{1 - tq^{j-i}}{1 - q^{j-i}}.$$

Similarly for cylindric partitions, using (6.14) together with the following proposition we obtain Theorem 8.

Proposition 7.

$$(6.16) \quad Z_{\text{cyl}}(R^-, R^+) = \prod_{n=1}^{\infty} \frac{1}{1 - q^{nT}} \prod_{k,l=1}^T H(q^{(k-l)(T)+(n-1)T} R_k^-, R_l^+),$$

where $i_{(T)}$ is the smallest positive integer such that $i \equiv i_{(T)} \pmod{T}$.

Proof. We use

$$(6.17) \quad \sum_{\lambda} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) = H(x, y) \sum_{\tau} Q_{\nu/\tau}(x) P_{\mu/\tau}(y).$$

The proof of this is analogous to the proof of Proposition 5.1 that appeared in our earlier paper [V1]. Also, see Example 26 of Chapter I, Section 5 of [Mac].

The proof of (6.16) uses the same idea as in the proof of Proposition 1.1 of [Bo].

We start with the definition of $Z_{\text{cyl}}(R^-, R^+)$. We omit index cyl.

$$\begin{aligned} Z(R^-, R^+) &= \sum_{\Lambda, M} W(\Lambda, M, R^-, R^+) = \sum_{\Lambda, M} q^{|\Lambda|} \prod_{n=1}^T P_{\lambda^{n-1}/\mu^n}(R_n^-) Q_{\lambda^n/\mu^n}(R_n^+) \\ &= \sum_{\Lambda, M} q^{|\mu|} \prod_{n=1}^T P_{\lambda^{n-1}/\mu^n}(qR_n^-) Q_{\lambda^n/\mu^n}(R_n^+) \end{aligned}$$

If $x = (x_1, x_2, \dots, x_T)$ is a vector we define the shift as $\text{sh}(x) = (x_2, \dots, x_T, x_1)$. We set $R_0^\pm = R_T^\pm$, $\mu_0 = \mu_T$ and $\tau_0 = \tau_T$. If using the formula (6.17) we substitute sums over λ^i 's with sums over τ^{i-1} 's

we obtain

$$\begin{aligned}
Z(R^-, R^+) &= H(q \operatorname{sh} R^-; R^+) \sum_{\mu, \tau} q^{|\mu|} Q_{\mu^1/\tau^0}(R_T^+), P_{\mu^0/\tau^0}(qR_1^-) \cdot \\
&\quad \cdot Q_{\mu^2/\tau^1}(R_1^+) P_{\mu^1/\tau^1}(qR_2^-) \cdots Q_{\mu^0/\tau^{T-1}}(qR_{T-1}^+) P_{\mu^{T-1}/\tau^{T-1}}(R_T^-) \\
&= H(q \operatorname{sh} R^-; R^+) \sum_{\mu, \tau} W(\operatorname{sh} \mu, \tau, q \operatorname{sh} R^-, R^+) \\
&= H(q \operatorname{sh} R^-; R^+) Z(q \operatorname{sh} R^-, R^+).
\end{aligned}$$

Since $\operatorname{sh}^T = \operatorname{id}$ if we apply the same trick T times we obtain

$$\begin{aligned}
Z(R^-, R^+) &= \prod_{i=1}^T H(q^i \operatorname{sh}^i R^-; R^+) \cdot Z(q^T R^-, R^+) \\
&= \prod_{i=1}^T H(q^i \operatorname{sh}^i R^-; R^+) \cdot Z(sR^-, R^+).
\end{aligned}$$

where $s = q^T$. Thus,

$$Z = \prod_{n=1}^{\infty} \prod_{i=1}^T H(q^{i+(n-1)T} \operatorname{sh}^i R^-; R^+) \lim_{n \rightarrow \infty} Z(s^n R^-, R^+).$$

Because

$$\lim_{n \rightarrow \infty} Z(s^n R^-, R^+) = \lim_{n \rightarrow \infty} Z(\operatorname{trivial}, R^+) = \prod_{n=1}^{\infty} \frac{1}{1 - s^n}$$

and

$$\begin{aligned}
\prod_{i=1}^T H(q^i \operatorname{sh}^i R^-, R^+) &= \prod_{l=1}^T \left[\prod_{k=l+1}^T H(q^{k-l} R_k^-, R_l^+) \prod_{k=1}^l H(q^{T+k-l} R_k^-, R_l^+) \right] \\
&= \prod_{k, l=1}^T H(q^{(k-l)(T)} R_k^-, R_l^+),
\end{aligned}$$

we conclude that (6.16) holds. \square

Observe that if in Theorem 8 we let $T \rightarrow \infty$, i.e. the circumference of cylinder go to infinity then we reconstruct (6.15).

7. CONCLUDING REMARKS

In this paper, we determine the generating functions of plane overpartitions with several types of constraints. In particular, we can compute the generating function of plane overpartitions with at most r rows and c columns and the generating function of plane overpartitions with entries at most n . The natural question is therefore to put

those constraints together and to compute the generating function of plane overpartitions with at most r rows, c columns and entries at most n . Unfortunately this generating function is not a product.

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[†] CNRS, LRI, UNIVERSITÉ PARIS-SUD, 91405 ORSAY CEDEX, FRANCE

^{*} MATHEMATICS 253-37, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA